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ON CLASS NUMBER of a GALOIS EXTENSION

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Let K be a finite Galois extension of an algebraic number field k . The central extension $\hat{K}$ and the genus field K^* over k are defined in [1], [3]. The central class number $z_{K/k} = (\hat{K} : K)$ and the genus number $g_{K/k} = (K^* : K)$ of K with respect to k are given in [1], [3].

Let $a_{K/k}$ be the ambiguous ideal class number of K respect to k . If K / k is a cyclic extension, or a Galois extension of a prime power degree, we have some relations between the class number h_K of K and $a_{K/k}$ ([2], [3], [5]).

In [4], it is proved:

"Let K / k be a cyclic extension of a prime power degree ℓ^ν , and suppose K and the absolute class field $\bar{k}$ of k are disjoint over k, i.e. $K \cap \bar{k} = k$. Then h_K is prime to ℓ if and only if $a_{K/k} = h_K$ and h_k is prime to ℓ ."

We shall generalize this criterion to a Galois extension of a prime power degree. In this note, h_k and $\bar{k}$ for a finite algebraic number field k mean as above.

PROPOSITION. Let k be an algebraic number field and K / k be a Galois extension of degree n . Suppose $K \cap \bar{k} = k$.

(1) If h_K is prime to n , then $z_{K/k} = g_{K/k} = a_{K/k} = h_k$, and h_k is prime to n .

(2) Let $n = \ell^\nu$ be a prime power. Then h_K is prime to ℓ if and only if $z_{K/k} = g_{K/k} = a_{K/k} = h_k$ and h_k is prime to ℓ .

Proof. (1) Since $z_{K/k} = (\hat{K} : K^*) \cdot g_{K/k}$, $g_{K/k} = (K^* : K \cdot \bar{k}) h_k$, h_K , $z_{K/k}$ and $g_{K/k}$ are divisible by h_k . In the decomposition

$$\frac{h_K}{h_k} = \frac{g_{K/k}}{h_k} \frac{h_K}{g_{K/k}} = \frac{z_{K/k}}{h_k} \frac{h_K}{z_{K/k}}$$

if h_K is prime to n , then $\dfrac{g_{K/k}}{h_k}$, $\dfrac{z_{K/k}}{h_k}$ are prime to n .

From the genus number formula and the central class number formula, we have

$$\frac{g_{K/k}}{h_k} = \frac{T_{\mathfrak{p}} T e'_{\mathfrak{z}}}{(K_0 : k)(E_k : E_k \cap N_{K/k} U_K)},$$

$$\frac{z_{K/k}}{h_k} = \frac{T_{\mathfrak{z}} T e'_{\mathfrak{z}} \cdot (k^\times \cap N_{K/k} J_K : N_{K/k} K^\times)}{(K_0 : k)\ (E_k : E_k \cap N_{K/k} K^\times)}$$

(notations in these formulas are defined in [1], [3]).

$$E_k \cap N_{K/k} U_K \supset E_k^n = \{ \varepsilon^n = N_{K/k} \varepsilon \mid \varepsilon \in E_k \},$$

$$E_k \cap N_{K/k} K^\times \supset E_k^n$$

and it is well-known

$$(k^\times \cap N_{K/k} J_K) / N_{K/k} K^\times \cong H^{-3}(G, \mathbb{Z}) / F$$

where G is a Galois group of K over k and F is the subgroup of TATE cohomology group $H^{-3}(G, \mathbb{Z})$, generated by $\mathrm{Inj}_{G_{\mathfrak{z}}/G} H^{-3}(G_{\mathfrak{z}}, \mathbb{Z})$ for all the infinite and the finite prime divisors $\mathfrak{z}$ and its decomposition group $G_{\mathfrak{z}}$.

Therefore, the prime factors of $\dfrac{g_{K/k}}{h_k}$ and $\dfrac{z_{K/k}}{h_k}$ are those of n , also

$\dfrac{g_{K/k}}{h_k} = \dfrac{z_{K/k}}{h_k} = 1$; $z_{K/k} = g_{K/k} = h_k$.

Let I_K be the ideal group of K and P_K be the principal ideal group of K . Then

$$A = \{ \mathfrak{a} \in I_K \mid \mathfrak{a}^{1-\sigma} \in P_K \text{ for any } \sigma \in G \},$$

$$H = \{ \mathfrak{a} \in I_K \mid N_{K/k}\mathfrak{a} \in P_K \}$$

are the subgroups of I_K , and

$$A \cap H \subset \{ \mathfrak{a} \in I_K \mid \mathfrak{a}^{n} \in P_K \}.$$

It follows for the ambiguous ideal class number $a_{K/k}$ of K respect to k

$$a_{K/k} = (A : P_K) \doteq (A : A \cap H)(A \cap H : P_K),$$

where the prime factors of $(A \cap H : P_K)$ are also those of n . If h_K is prime to n , then $a_{K/k}$ is prime to n , hence

$$(A \cap H : P_K) = 1; \quad A \cap H = P_K .$$

By $g_{K/k} = h_k$, $K^* = K \cdot \bar{k}$ is the genus field of K over k , i.e. the class field corresponding to H over K . So we have

$$g_{K/k} = (K^* : K) = (I_K : H) = (I_K : AH)(AH : H)$$
$$= (I_K : AH)(A : A \cap H) = (I_K : AH) \cdot a_{K/k} .$$

On the other hand, the number $a^{(0)}_{K/k}$ of ideal classes represented by an ambiguous ideal in K / k is given by [5]:

$$a^{(0)}_{K/k} = \frac{h_k \cdot \prod_{g} e_{g}}{|H^1(G, E_K)|} .$$

If h_K is prime to n , then $a_{K/k}, a^{(0)}_{K/k}$ and h_k are prime to n , that is, $a^{(0)}_{K/k} = h_k$. Since $g_{K/k} = h_k \equiv 0 \bmod a_{K/k}$, and $a_{K/k} \equiv 0 \bmod a^{(0)}_{K/k}$, hence $g_{K/k} = h_k = a_{K/k}$.

(2) In case of $n = l^{\nu}$, we know in [3]

$$h_K \equiv z_{K/k} \bmod l .$$

References

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